

An Adaptive Meshfree Least-Squares Method

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Abstract

In this paper, an adaptive meshfree least-squares method (LSM) is proposed, which does not use any mesh predefined through node connectivity. In this present formulation, a radial point collocation procedure is used to discretize the system governing equations. A modified least-squares technique is employed to stabilize the solution to obtain more stable and accurate results. The adaptivity scheme adopted in this work uses an error indicator based on interpolation error. Voronoi diagram is used in the refinement procedure at each adaptive step for additional node insertion. Numerical examples are presented to demonstrate that the proposed adaptive meshfree LSM can obtain efficiently stable solutions of desired accuracy.

Introduction

Finite Element Method (FEM) has been now widely used to solve various kinds of engineering problems. Although FEM has achieved great success, it has encountered mesh-related difficulties while dealing with large deformation problems due to large mesh distortion [2, 3]. In the last few decades, a new generation of numerical methods, meshfree method, has been developed to overcome those mesh-related difficulties. Meshfree methods are formulated

based on a set of scattered nodes and mesh-related difficulties are avoided as no mesh is used. This attractive feature also facilitates meshfree method to couple with adaptive techniques to perform adaptive analysis, because the nodes can be removed or inserted easily. Comprehensive reviews of the recent development of meshfree methods can be found in the Refs. [1, 2].

Based on the formulation, meshfree methods can be categorized into three major categories: meshfree method based on strong-form (or short for meshfree strong-form methods), meshfree method based on weak-form [2, 3] (or short for meshfree weak-form methods) and meshfree method based on both strong-form and weak-form [4, 5] (or short for meshfree weak-strong form methods). Among the three categories, meshfree strong-form methods are the simplest and most straightforward method. Neither complicated formulation nor integration is involved. As the reason, meshfree strong-form method can potentially couple with adaptive scheme to provide an adaptive meshfree method. Although strong-form methods are suffering from instability and low accuracy [10, 11, 12], technique can be developed to provide a stable result with higher accuracy. Some relevant works regarding adaptive meshfree strong-form methods have been done and can be found in the literature [6, 8].

In this paper, a least-squares method (LSM) is proposed for adaptive analysis. A radial point collocation procedure [9] is used to discretize the system governing equations. A modified least-squares technique is used to stabilise the solution. An elastostatics problem and a Poisson's equation problem are used to validate the proposed method. Numerical examples show that the adaptive meshfree LSM proposed in this work can provide stable solutions of desired accuracy efficiently.

Radial Point Collocation Method

The idea of well-known conventional collocation method [9] is first to approximate the local unknown variable through interpolation of the nodal unknown variable at the surrounding nodes. The partial differential equations (PDEs) that govern the problem can then be discretized for each node using this approximation to obtain a set of algebraic equations.

In this work, a radial point collocation method (RPCM) is used, in which radial basis functions (RBFs) augmented with polynomial functions are used to construct shape functions. Unknown field variable is first interpolated through those at the nodes in the support domain as follows,

$$u^h(\mathbf{x}, \mathbf{x}_Q) = \sum_i^n R_i(\mathbf{x})a_i(\mathbf{x}_Q) + \sum_1^m p_i(\mathbf{x})b_i(\mathbf{x}_Q) = \mathbf{R}^T(\mathbf{x})\mathbf{a}(\mathbf{x}_Q) + \mathbf{p}^T(\mathbf{x})\mathbf{b}(\mathbf{x}_Q) \quad (1)$$

where n is number of node in the support domain, m is the number of terms of monomials (polynomial basis). $\mathbf{a}(\mathbf{x}_Q)$ is the vector of coefficients of radial basis $\mathbf{R}^T(\mathbf{x})$, $\mathbf{b}(\mathbf{x}_Q)$ is the vector of coefficients of polynomial basis $\mathbf{p}^T(\mathbf{x})$.

In this paper, Multi-quadratic (MQ) is used as a RBF:

$$R_i(x, y) = \left(r_i^2 + (\alpha_c d_c)^2 \right)^q \quad (2)$$

where r_i is a distance between the interpolation point at \mathbf{x} and a neighbourhood node at \mathbf{x}_i

$$r_i = \sqrt{(x - x_i)^2 + (y - y_i)^2} \quad (3)$$

and d_c is a characteristic length which is the average nodal spacing in the support domain. In Eq. (3), α_c and q are the shape parameters for MQ-RBF. The detailed definition of support domain and the choice of shape parameter can be found in Ref. [3].

Follow the standard procedure described in Section 5.7 of Ref. [3], unknown variable can be approximated as

$$\mathbf{u}^h(\mathbf{x}) = \mathbf{\Phi}(\mathbf{x})\mathbf{U}_S \quad (4)$$

where $\mathbf{\Phi}$ is the matrix of shape functions [3].

The derivative of the unknown variable can be obtained easily using Eq. (4). For example, the first derivative can be obtained as

$$\mathbf{u}^{h,k}(\mathbf{x}) = \mathbf{\Phi}_{,k}(\mathbf{x})\mathbf{U}_S \quad (5)$$

The governing equation and Neumann boundary condition can now be discretized at each node using Eq. (4) by simple collocation. A set of algebraic equations can be assembled to obtain the following form,

$$\mathbf{K}_G\mathbf{U} - \mathbf{F}_G = 0 \quad (6)$$

where \mathbf{K}_G denotes stiffness matrix, \mathbf{U} is the vector of the unknown variable at all nodes in the problem domain, and \mathbf{F}_G is the vector of forces applied at all nodes in the problem domain.

Unknown variable can finally be solved by solving the resultant set of algebraic equations after the essential boundary condition is imposed. The advantage of the collocation method is that it's simple and easy to be implemented. However, instability is often encountered in conventional collocation methods [9, 11, 12]. In this paper, a modified least-squares technique is proposed to stabilize RPCM to obtain more stable and accurate results

A Modified Least-Squares Procedure

We define a functional II in the form of

$$\mathit{II} = \{\mathbf{F}_G - \mathbf{K}_G\mathbf{U}\}^T \{\mathbf{F}_G - \mathbf{K}_G\mathbf{U}\} \quad (7)$$

where II is the L2 error norm of the residual in Eq. (6).

In seeking the minimal of the functional Π , we have

$$\frac{\partial \Pi}{\partial \mathbf{U}} = -2\mathbf{K}_G^T \{\mathbf{F}_G - \mathbf{K}_G \mathbf{U}\} = 0 \quad (8)$$

which gives

$$\mathbf{K}_G^T \mathbf{F}_G = \mathbf{K}_G^T \mathbf{K}_G \mathbf{U}$$

or

$$\hat{\mathbf{F}} = \hat{\mathbf{K}} \mathbf{U} \quad (9)$$

where $\hat{\mathbf{K}} = \mathbf{K}_G^T \mathbf{K}_G$ is the modified stiffness matrix and $\hat{\mathbf{F}} = \mathbf{K}_G^T \mathbf{F}_G$ is the modified force vector.

Note that in the present least-squares procedure, Neumann boundary condition is imposed in the process of forming the stiffness matrix \mathbf{K}_G and force vector \mathbf{F}_G . Essential boundary condition is imposed at the final stage after obtaining Eq. (9). This procedure is a little different compared to the standard least-squares method [10] procedure where the least-squares operation is performed after all the boundary conditions are imposed in Eq. (6).

It is clear that the modified stiffness matrix $\hat{\mathbf{K}}$, is usually a symmetric matrix and positive definite after the imposition of the essential boundary condition. A more stable and accurate result can be obtained using a standard linear equation solver to solve Eq. (9), such as the Cholesky solver.

Adaptive Procedure

As linear and static problems are studied in this work, only node refinement is involved in the

adaptive procedure. Adaptive algorithm based on interpolation error proposed by Behrens et al [6] is adopted here. Error indicator used for the node refinement is defined as

$$\eta(x_i) = \left| u^S(x_i) - u^{\bar{S}}(x_i) \right| \quad (10)$$

where $u^S(x_i)$ is the value of a field variable or function of field variable such as stress at node x_i ; $u^{\bar{S}}(x_i)$ is the field variable or function of field variable obtained by an interpolation using the vicinity nodes in the support domain of node x_i excluding node x_i , i.e. $\bar{S} = S \setminus \{x_i\}$, where S is the node set in the support domain of node x_i . $\eta(x_i)$ reflects the local reproduction quality of the interpolation around node x_i . Note that $\eta(x_i)$ will vanish if the field variable is linear around node x_i .

The hypothesis making here is that if the obtained result is a good approximation, the difference between $u^N(x_i)$ and $u^{\bar{N}}(x_i)$ is very small. Thus, no refinement is required to perform around the node x_i . In contrast, if value of error indicator is very large, it indicates that quality of reproduction is not good enough. Thus, refinement scheme is executed to insert additional nodes around node x_i .

Critical criteria for refinement are defined as

$$\eta(x_i) > \kappa_1 \eta^* \quad \text{and} \quad \eta^* > \kappa_2 u_{\max} \quad (11)$$

where κ_1 and κ_2 are the tolerant value which satisfy $0 < \kappa < 1$. η^* is the maximum error indicator value in the entire problem domain and u_{\max} is the maximum field variable or function of field variable. Node refinement will be executed if the critical criteria for refinement are satisfied.

Since no mesh is used in the meshfree method, the problem domain can be refined conveniently by node insertion. In this work, voronoi diagram is used to locate the position for the additional nodes, as shown in Fig. (1).

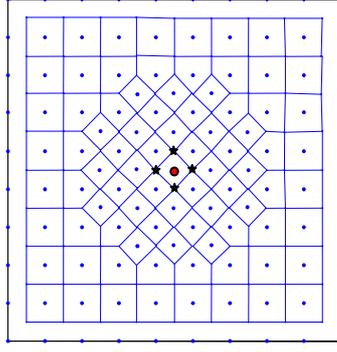


Figure 1. Refinement based on voronoi diagram

Numerical Example

In this paper, the validity of the proposed adaptive analysis is tested on two example problems. The first example is a Poisson's equation problem with three essential boundary conditions and one derivative boundary condition given. The second example is a 2-D cantilever beam subjected to a load at the free end.

Example 1:

In example 1, a Poisson's equation is considered as below.

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = \sin \pi x \sin \pi y \quad (12)$$

The problem domain is $\Omega = [0,1] \times [0,1]$, with Neumann boundary condition

$$\frac{\partial u}{\partial x} = \frac{1}{2\pi} \sin \pi y \quad \text{along } x = 1 \quad (13)$$

and essential boundary condition

$$u = -\frac{1}{2\pi^2} \sin \pi x \sin \pi y \quad \text{along } x = 0, y = 0 \quad \text{and } y = 1 \quad (14)$$

The adaptive analysis starts with regular distribution of 6×6 nodes at initial step. Tolerant values for critical error indicator are $\kappa_1 = \kappa_2 = 0.01$.

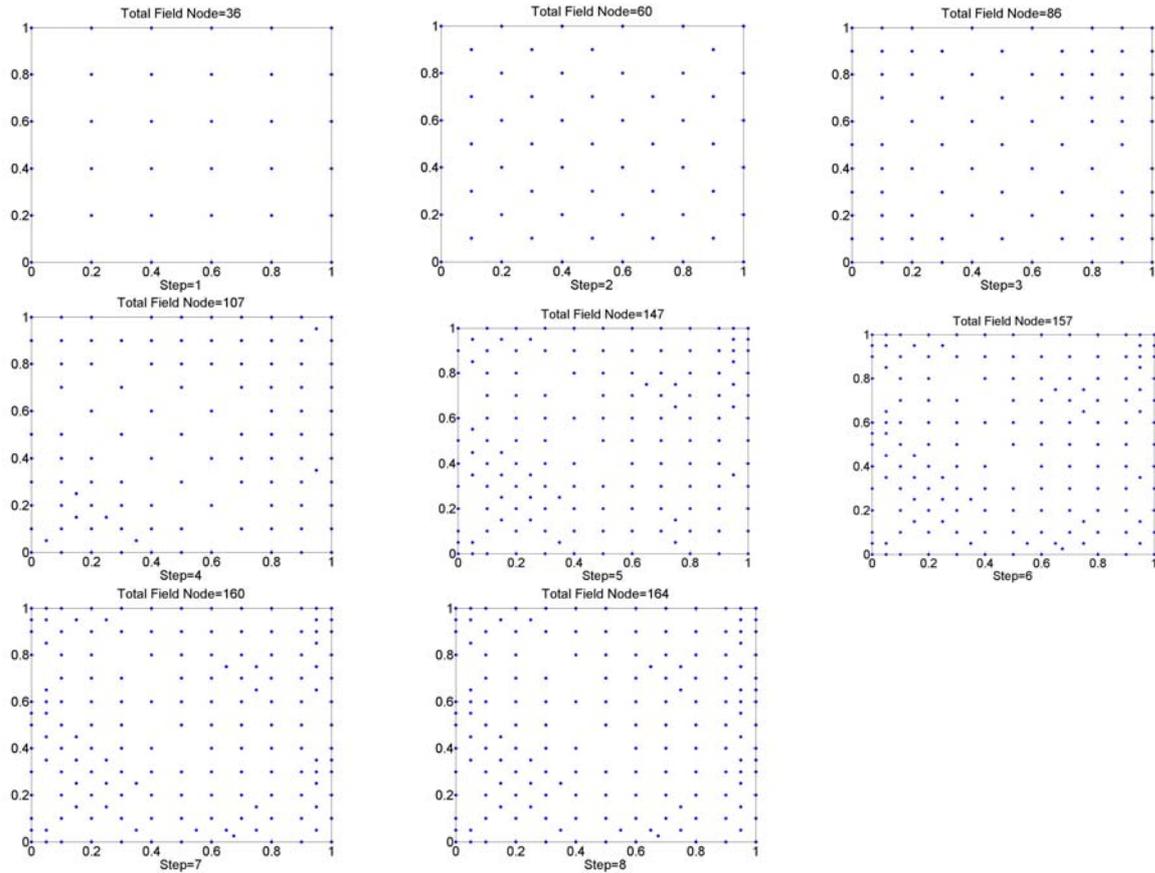


Figure 2. Node distribution at each adaptive step for Poisson's equation

Starting with 36 regular distributed nodes, the computation ends at 8th step with 164 nodes, the overall error norm of field variable u has been improved from 4.17% to 0.45%.

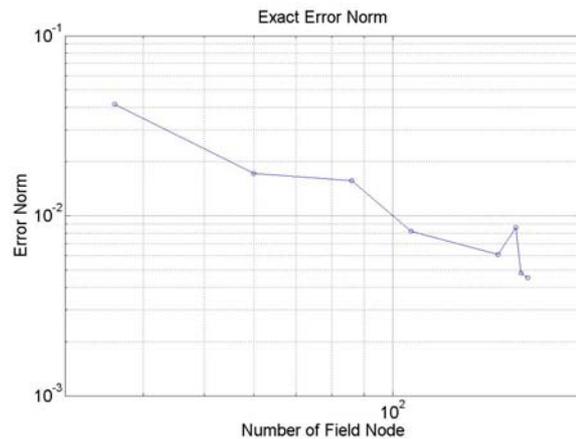


Figure 3. Error norm of solution for Poisson's equation at each step

Example 2:

In the second problem, a plane stress problem is considered. A cantilever beam of 48m by 12m and 1m thick is subjected to a traction force of 1000N at the free end. The material properties

are Young's modulus $E = 3 \times 10^7 \text{ N/m}^2$ and Poisson ratio $\nu = 0.3$. The exact solution is given by Timonshenko and Goodier [7].

The adaptive analysis starts with regular distribution of 5×11 nodes at initial step. Effective stress is used to determine the value of error indicator and the tolerant values for critical error indicator are $\kappa_1 = 0.05$ and $\kappa_2 = 0.0075$.

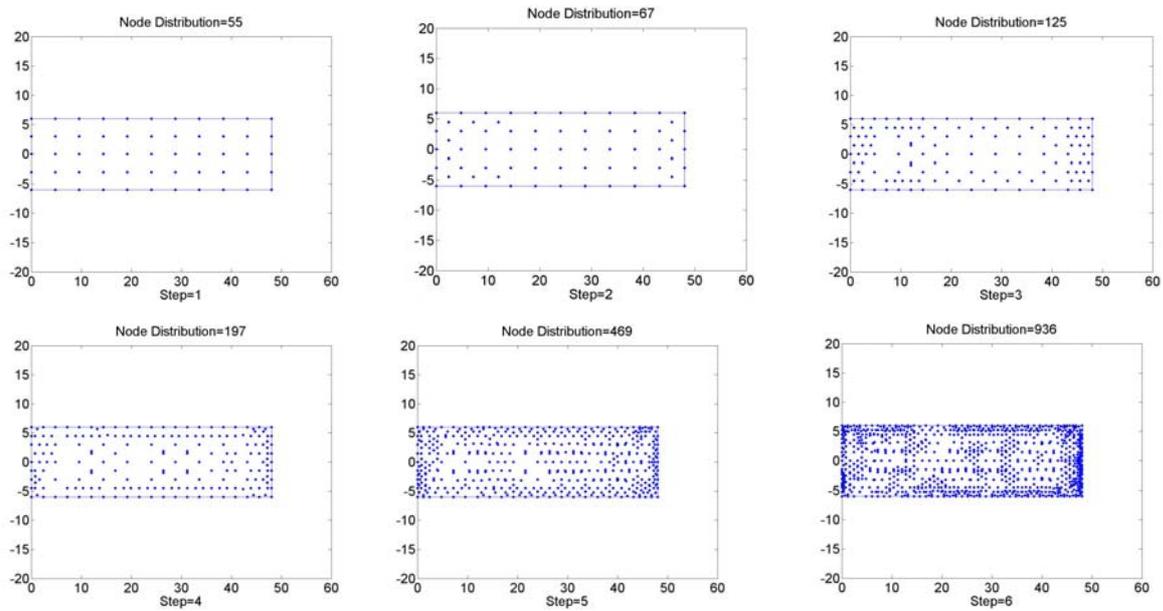


Figure 4. Node distribution of Cantilever beam at each adaptive step

Starting with 55 regular distributed nodes, the computation ends at 6th step with 936 nodes, the error norm of effective stress has been improved from 15.55% to 0.81%.

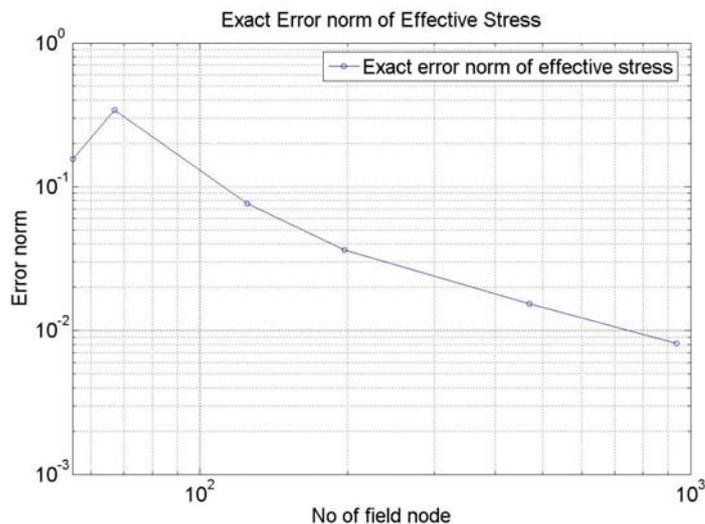


Figure 5. Error norm of effective stress at each step.

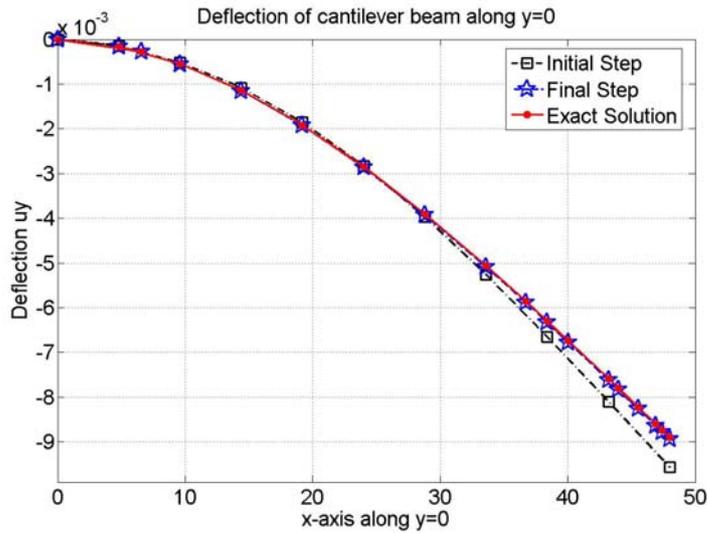


Figure 6. Deflection of cantilever beam along $y = 0$ at first step and final step.

Conclusion

From the results of the two numerical examples, it concludes that the proposed adaptive LSM has obtained high accuracy results efficiently. Good performance of adaptive scheme through error indicator based on interpolation error is observed. The modified least-squares technique has also provided a stable result at each adaptive step, which made the adaptive operation possible. As meshfree method allows inserting node easily in the problem domain, it makes the implementation of refinement process much simpler. The attractive advantages of meshfree method have been putting across in the adaptive analysis well and have been demonstrated clearly in this paper.

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